

QUANTIZATIONS OF MOMENTUM MAPS AND G -SYSTEMS

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ABSTRACT. In this note, we give an explicit formula for a family of deformation quantizations for the momentum map associated with the cotangent lift of a Lie group action on \mathbb{R}^d . This family of quantizations is parametrized by the formal G -systems introduced in [10] and allows us to obtain classical invariant Hamiltonians that quantize without anomalies with respect to the quantizations of the action prescribed by the formal G -systems.

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1. INTRODUCTION

The concept of momentum map plays a fundamental role in the classical description of hamiltonian dynamical systems (in finite and in infinite dimension), see for example [14]. The Marsden-Weinstein reduction procedure on momentum map level sets (with all of its various generalizations) is a powerful method to study dynamical systems with symmetries and to construct new symplectic (Poisson, Kähler, hyper-Kähler and so on) manifolds from old ones endowed with a Lie group action preserving the relevant geometric structures.

The quantum counterparts of momentum maps (which are special deformation quantizations, introduced by Ping Xu in [25], of classical momentum maps regarded as a Poisson maps) and the corresponding reduction procedure should play a similar fundamental role in the study of quantum systems with symmetries (see for example

[12], [24], [25], [19] and also the monograph [20]). However, one of the difficulty in the theory of quantum momentum maps is that explicit formulas are hard to come by.

In this paper, we give a such an explicit formula for a family of deformation quantizations for the momentum map J associated with the cotangent lift $\tilde{\varphi}$ of an action φ of Lie group G on \mathbb{R}^d (Theorem 10). The result is a family of deformation quantizations (i.e unital algebra morphisms), parametrized by the G -systems introduced in [10], from the Gutt star-algebra to the standard star-algebra on the cotangent bundle:

$$(1.1) \quad J^a : (C^\infty(\mathcal{G}^*)[[\hbar]], \star_G) \longrightarrow (C^\infty(T^*\mathbb{R}^d)[[\hbar]], \star_{st})$$

where a is a formal G -system, that is, a Maurer-Cartan element in a certain differential graded algebra of formal amplitudes associated with the action (see Section 2.1 for a short reminder).

These quantizations do not satisfy in general the additional conditions defining quantum momentum maps as described in [25] (i.e. that the star-product on the range of (1.1) should be equivariant with respect to the representation by pullbacks of the cotangent lift action and that $\frac{i}{\hbar}[J^a(v), f] = \tilde{X}^v(f)$ must hold for all $f \in C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ and $v \in \mathcal{G}$, where \tilde{X}^v is the fundamental vector field of the cotangent lift action) but rather deformations of these conditions, controlled by formal G -systems (Proposition 13 and Theorem 14).

These deformed conditions can be understood in terms of the quantizations introduced in [10]: Namely a formal G -system a associated with an action of a Lie group G on \mathbb{R}^d produces a representation of G by formal operators T_g^a (obtained as asymptotic expansions of certain Fourier integral operators whose amplitudes are given by the G -system a) on the space $C^\infty(\mathbb{R}^d)[[\hbar]]$ of formal functions on \mathbb{R}^d (playing the role of the quantum Hilbert space $L^2(\mathbb{R}^d)$ of states in the formal setting). This quantization T^a lifts to the space $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ of formal functions on the cotangent bundle (playing the role of the quantum algebra of observables in the formal setting), producing a representation \tilde{T}^a of G on this space that deforms the representation obtained by pullbacks of the cotangent lift action.

The standard star-product is always equivariant with respect to the deformed cotangent lift representation \tilde{T}^a (Proposition 13). Moreover the deformed condition

$$\frac{i}{\hbar}[J^a(v), f] = \tilde{t}_v^a f = \tilde{X}^v(f) + \mathcal{O}(\hbar),$$

holds for all $f \in C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ and $v \in \mathcal{G}$, where \tilde{t}_v^a is the derivative of the lifted representation \tilde{T}^a at the group unit.

As a by-product, we obtain a family of invariant classical Hamiltonians $H_f^a = J^a f$, where f is in the center of $(C^\infty(\mathcal{G}^*), \{, \})$. These invariant Hamiltonians quantize without anomalies with respect to the action quantization T^a (Theorem 33), i.e., the quantum Hamiltonians \hat{H}_f^a are invariant with respect to the quantized action: $T_g^a \hat{H}_f^a T_{g^{-1}}^a = \hat{H}_f^a$, where \hat{H}_f^a is the standard quantization of H_f^a by pseudo-differential operators.

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2. SETTING AND RESULTS

In this section, we review how to quantize an action of a Lie group G on \mathbb{R}^d using the G -systems introduced in [10]. They are Maurer-Cartan elements in a certain differential graded algebra of amplitudes. Then we recall the notion of quantum momentum maps as defined in [25], and we conclude with a presentation of our main results: Namely, Theorem 10 gives a family of deformation quantization of the momentum map associated with the cotangent lift of an action on \mathbb{R}^d and Proposition 13 and Theorem 14 explain how these quantizations satisfy a deformed version of Ping Xu's original definition.

2.1. Quantization of symmetries and G -systems. In [10], we introduced a Differential Graded Algebra (DGA) of amplitudes $(\mathcal{A}_\varphi^\bullet, d, \star)$ associated with a bounded action φ of a Lie group G on \mathbb{R}^d . By bounded action, we mean that $\varphi_g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a smooth map for each $g \in G$ such that $|\partial^\alpha \varphi_g^i(x)|$ ($i = 1, \dots, d$) is uniformly bounded for each multi-index $\alpha \in \mathbb{N}^d \setminus \{0\}$. For each positive integer k , we set

$$\mathcal{A}_\varphi^k := \left\{ a : G^k \rightarrow S_{2d}(1) \right\},$$

where $S_{2d}(1)$ is the space of bounded amplitudes, i.e., families of smooth functions on $T^*\mathbb{R}^d$ depending on a parameter $\hbar \in [0, \hbar_0]$ uniformly bounded on $T^*\mathbb{R}^d \times [0, \hbar_0]$ as well as all their derivatives (see [21] for more details on amplitude spaces). The differential $d : \mathcal{A}_\varphi^k \rightarrow \mathcal{A}_\varphi^{k+1}$ is given by

$$(da)_{g_1, \dots, g_{k+1}} = \sum_{i=1}^k (-1)^i a_{g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}}.$$

The associative product $\star : \mathcal{A}_\varphi^k \times \mathcal{A}_\varphi^l \rightarrow \mathcal{A}_\varphi^{k+l}$ is induced from the composition $T^a \circ T^b = T^{a \star b}$ of certain Fourier Integral Operators (FIO) acting on $L^2(T^*\mathbb{R}^d)$ associated with amplitudes in $\mathcal{A}_\varphi^\bullet$; namely, for $a \in \mathcal{A}_\varphi^k$, we have

$$T_{g_1, \dots, g_k}^a \psi(x) := \int \psi(\bar{x}) a_{g_1, \dots, g_k}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{(g_1 \dots g_k)^{-1}}(x) - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^d}.$$

Definition 1. A G -system is a Maurer-Cartan element in $\mathcal{A}_\varphi^\bullet$, i.e., an element $a \in \mathcal{A}_\varphi^1$ satisfying the Maurer-Cartan equation $da + a \star a = 0$.

As explained in [10], a G -system a produces a representation of G on $L^2(T^*\mathbb{R}^d)$. This representation quantizes the cotangent lift $\tilde{\varphi}$ of the action on $T^*\mathbb{R}^d$ in the sense of semi-classical analysis (for instance, see symplectomorphism quantization in [11, 21]), as we shall see in the next proposition:

Proposition 2. *Let f be a suitably bounded smooth function (e.g. uniformly bounded by a polynomial in ξ) and a be G -system associated with an action φ of a Lie group G on \mathbb{R}^d . Then we have*

$$(2.1) \quad T_g^a \text{Op}(f) T_{g^{-1}}^a = \text{Op}(\tilde{T}_g^a f), \quad \tilde{T}_g^a f = \tilde{\varphi}_g^* f + \mathcal{O}(\hbar),$$

where $\text{Op}(f)$ is the **standard quantization** of f by pseudo-differential operators (see [11, 21] for more details): i.e.,

$$(2.2) \quad \text{Op}(f)\psi(x) := \int \psi(\bar{x})f(x, \bar{\xi})e^{\frac{i}{\hbar}\langle \bar{\xi}, x - \bar{x} \rangle} \frac{d\bar{x}d\bar{\xi}}{(2\pi\hbar)^d}.$$

Proof. A direct computation shows that $T_g^a \text{Op}(f) T_{g^{-1}}^a = \text{Op}(\tilde{T}_g^a f)$, where

$$(\tilde{T}_g^a f)(x, \xi) = \int a_g(x, \bar{\xi})a_{g^{-1}}(\tilde{x}, \xi)f(\bar{x}, \tilde{\xi})e^{\frac{i}{\hbar}S_{x, \xi}(\bar{x}, \bar{\xi}, \tilde{x}, \tilde{\xi})} \frac{d\bar{x}d\bar{\xi}d\tilde{x}d\tilde{\xi}}{(2\pi\hbar)^{2d}},$$

with the phase $S_{x, \xi}$ given by

$$S_{x, \xi}(\bar{x}, \bar{\xi}, \tilde{x}, \tilde{\xi}) = \bar{\xi}(\varphi_{g^{-1}}(x) - \bar{x}) + \tilde{\xi}(\bar{x} - \tilde{x}) + \xi(\varphi_g(\tilde{x}) - x).$$

Computing the critical point of $S_{x, \xi}$ w.r.t. the integration variables, we get

$$(2.3) \quad \bar{x} = \tilde{x} = \varphi_{g^{-1}}(x), \quad \bar{\xi} = \tilde{\xi} = (T_x^* \varphi_g)\xi.$$

Using the stationary phase theorem, we get the first term of the asymptotic expansion of \tilde{T}_g^a ; namely,

$$(\tilde{T}_g^a f)(x, \xi) = a_g(x, (T_x^* \varphi_g)\xi)a_{g^{-1}}(\varphi_{g^{-1}}(x), \xi)f(\varphi_{g^{-1}}(x), (T_x^* \varphi_g)\xi) + \mathcal{O}(\hbar).$$

Now we prove that

$$(2.4) \quad a_g(x, (T_x^* \varphi_g)\xi)a_{g^{-1}}(\varphi_{g^{-1}}(x), \xi) = 1$$

by doing a similar computation using the relation $T_g^a T_{g^{-1}}^a = \text{id}$, which holds because T^a is a representation as shown in [10]. Namely, a direct calculation yields $T_g^a T_{g^{-1}}^a = \text{Op}(g)$, where

$$(2.5) \quad g(x, \xi) = \int a_g(x, \bar{\xi})a_{g^{-1}}(\bar{x}, \xi)e^{\frac{i}{\hbar}F_{x, \xi}(\bar{x}, \bar{\xi})} \frac{d\bar{x}d\bar{\xi}}{(2\pi\hbar)^d},$$

with $F_{x, \xi}(\bar{x}, \bar{\xi}) = \bar{\xi}(\varphi_{g^{-1}}(x) - \bar{x}) + \xi(\varphi_g(\bar{x}) - x)$. By injectivity of Op , we have that $g = 1$. Using again, as above, the stationary phase theorem on (2.5), we obtain (2.4). \square

Standard quantization defines the **standard product** of (suitably bounded, see [11, 21]) smooth functions on the cotangent bundle: i.e.,

$$(2.6) \quad \text{Op}(f \star_{st} g) = \text{Op}(f) \circ \text{Op}(g),$$

Remark 3. Observe that both pseudo-differential operators (2.2) and the standard product (2.6) can be defined on the space $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ of formal power series in \hbar with coefficients in the smooth functions on $T^*\mathbb{R}^d$ by considering asymptotic expansions of both (2.2) and (2.6) in the limit $\hbar \rightarrow 0$ (again see [11, 21]). In particular, Equation (2.1) holds in this formal context, and \tilde{T}_g^a is now a formal operator on $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$.

In this paper, we will be mostly concerned with the formal version of $\mathcal{A}_\varphi^\bullet$, also introduced and discussed in more details in [10]. Let us briefly outline the construction. Instead of $\mathcal{A}_\varphi^\bullet$, we will consider the DGA $(\mathcal{P}_\varphi^\bullet, d, \star)$ of formal amplitudes. The construction is similar to the bounded amplitude case, and $\mathcal{P}_\varphi^\bullet$ can be regarded as

the asymptotics of $\mathcal{A}_\varphi^\bullet$ in the limit $\hbar \rightarrow 0$. The formal amplitudes in \mathcal{P}_φ^k are formal power series in \hbar of the form

$$a_{g_1, \dots, g_k}(x, \xi) = P_{g_1, \dots, g_k}^0(x) + \sum_{n \geq 1} \hbar^n P_{g_1, \dots, g_k}^n(x, \xi),$$

where $P_{g_1, \dots, g_k}^n(x, \xi)$ is a polynomial of degree at most n in ξ with coefficients in the smooth functions on \mathbb{R}^d . The corresponding operator T^a acts on the space $C^\infty(\mathbb{R}^d)[[\hbar]]$ of formal functions (i.e. formal power series in \hbar with coefficients in the smooth functions on \mathbb{R}^d):

$$T_{g_1, \dots, g_k}^a \psi(x) = P^0(x) \psi(\varphi_{(g_1 \dots g_k)^{-1}}(x)) + \sum_{n \geq 1} \hbar^n P(x, \frac{1}{i} \partial) \psi \Big|_{\varphi_{(g_1 \dots g_k)^{-1}}(x)}$$

Definition 4. A **formal G -system** is a Maurer-Cartan element in $\mathcal{P}_\varphi^\bullet$.

Similarly to the bounded case, a formal G -system a defines a formal representation T^a of G on $C^\infty(\mathbb{R}^d)[[\hbar]]$ deforming the representation by pullbacks,

$$(2.7) \quad T_g^a \psi(x) = P_0(x) \psi(\varphi_{g^{-1}}(x)) + \mathcal{O}(\hbar),$$

and a corresponding representation \tilde{T}^a of G on $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ defined by (2.1) (see also Remark 3), which deforms the representation by cotangent lift pullbacks,

$$(2.8) \quad \tilde{T}_g^a f(x) = \tilde{\varphi}_{g^{-1}}^* f + \mathcal{O}(\hbar),$$

where $\tilde{\varphi}_g$ is the cotangent lift of φ_g .

Remark 5. There always exists a formal G -system; namely, the trivial one: $a = 1$. For this formal G -system, the induced representation $T^{a=1}$ on $C^\infty(\mathbb{R}^d)[[\hbar]]$ is exactly the representation by pullbacks $\varphi_{g^{-1}}^*$ of the action. However, as exemplified by the Egorov Theorem (see [11, 21]), one will *not* have in general that the induced representation $\tilde{T}^{a=1}$ on $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ is the representation by pullbacks $\tilde{\varphi}_{g^{-1}}^*$ of the cotangent lift action, but rather, already in the trivial case, a deformation of it.

2.2. Quantum momentum maps. In this paragraph, we recall the notion of classical and quantum momentum maps (we refer the reader to [1, 3, 25] for more details).

Suppose we have an hamiltonian action φ of a Lie group G on a symplectic manifold M admitting a momentum map, which, in general, can be defined as a smooth map $J : M \rightarrow \mathcal{G}^*$ such that the hamiltonian vector field with hamiltonian $J^*(v)$, for $v \in \mathcal{G}$ seen as a linear function on \mathcal{G}^* , coincides with the fundamental vector field X^v . Moreover we require J to be equivariant with respect to symplectic action of G on the domain and the coadjoint action of G on the range.

This equivariance implies that J is a Poisson map from $T^*\mathbb{R}^d$ equipped with the symplectic Poisson bracket to \mathcal{G}^* equipped with the Kirillov-Kostant Poisson bracket. The associated pullback map is thus a Lie algebra morphism

$$J^* : (C^\infty(\mathcal{G}^*), \{, \}) \longrightarrow (C^\infty(M), \{, \}),$$

which, restricted to the linear functions on \mathcal{G}^* , yields a representation of the Lie algebra \mathcal{G} on the Lie algebra of classical observables, i.e. $(C^\infty(M), \{, \})$.

The quantum picture in deformation quantization starts by deforming the classical Lie algebra in the domain and range of J^* into quantum algebras. Then one deforms J^* into a unital algebra morphism between these quantum algebras, which,

similarly to the classical case, yields a representation of \mathcal{G} into the quantum algebra of observables quantizing the range.

Let us recall some basic definitions:

Definition 6. Let A be a Poisson manifold with Poisson bracket $\{, \}_A$. A **deformation quantization** of A is a **star-product** \star_A on $C^\infty(A)[[\epsilon]]$, i.e. a unital associative product (for which the constant function 1 is the unit) of the form

$$f \star_A g = fg + \sum_{n \geq 1} \epsilon^n B_n(f, g),$$

where the B_n 's are bidifferential operators such that the quantum commutator $[,]_\star$ is a deformation of the Poisson bracket: $\frac{1}{\epsilon}[f, g]_\star = \{f, g\} + \mathcal{O}(\epsilon)$. Observe that the formal parameter ϵ often is taken to be $\epsilon = \frac{\hbar}{i}$ in concrete example.

One natural choice for the quantization of the momentum map domain is the **Gutt star-product** \star_G . It comes from transporting the associative product on the universal enveloping algebra of \mathcal{G} to the polynomials on \mathcal{G}^* via the symmetrization map (see [15]). Another definition of this product is via the asymptotic expansion of a FIO (see [2] for instance), this is the definition we are going to use here: Let $f, g \in C^\infty(\mathcal{G}^*)[[\hbar]]$, then the Gutt star product $f \star_G g$ is the asymptotic expansion in the limit $\hbar \rightarrow 0$ of the integral:

$$(2.9) \quad (f \star_G g)(\theta) = \int f(\theta_1)g(\theta_2) e^{\frac{i}{\hbar} \langle \theta, \text{BCH}(v_1, v_2) - \langle \theta_1, v_1 \rangle - \langle \theta_2, v_2 \rangle \rangle} \frac{dv_1 d\theta_1 dv_2 d\theta_2}{(2\pi\hbar)^{\dim \mathcal{G}}},$$

where $\text{BCH}(v_1, v_2)$ is the BCH formula.

One good feature of this star-product is that for two Lie algebra elements $v, w \in \mathcal{G}$, which we regard as linear functions on \mathcal{G}^* , we have

$$\frac{i}{\hbar}[v, w]_{\star_G} = [v, w]_{\mathcal{G}}.$$

This property allows us to obtain representations of \mathcal{G} into the quantum algebra quantizing $(C^\infty(M), \{, \}_M)$ from classical momentum map deformation quantizations having for range the Gutt star-algebra.

Definition 7. Suppose we have two star-products \star_A and \star_B quantizing the Poisson algebras $(C^\infty(A), \{, \}_A)$ and $(C^\infty(B), \{, \}_B)$, respectively. A **deformation quantization** $\hat{\phi}$ from $(C^\infty(A)[[\epsilon]], \star_A)$ to $(C^\infty(B)[[\epsilon]], \star_B)$ of a **Poisson map** ϕ from B to A is a unital algebra morphism of the form

$$\hat{\phi}f = \phi^*f + \sum_{n \geq 1} \epsilon^n D_n(f),$$

where the D_n 's are differential operators. Again, in concrete examples, one often has $\epsilon = \frac{\hbar}{i}$.

In [25], Ping Xu introduced the notion of a quantum momentum map, which is a special deformation quantization of the classical momentum map regarded as Poisson map. Let us recall his definition, which involves the notion of quantum G -spaces.

Definition 8. (A version of Ping Xu's definition [25]). Let M be a symplectic manifold with a hamiltonian action φ of G on M admitting a momentum map J .

A star-product \star on $C^\infty(M)[[\hbar]]$ is **G -equivariant** if the pullback action φ^* acts on it by unital algebra morphisms. The data of a hamiltonian action together with a G -equivariant star-product as above is called a **quantum G -space**.

A **quantum momentum map**, quantizing J , is a deformation quantization of J having for domain the Gutt star-algebra and such that

- (1) its range $(C^\infty(M)[[\hbar]], \star)$ is a quantum G -space,
- (2) the condition $\frac{i}{\hbar}[Q(J)(v), f] = X^v(f)$ must hold for all $f \in C^\infty(M)[[\hbar]]$.

Remark 9. The original definition of a quantum momentum map is that of a unital algebra morphism from the universal enveloping algebra $\mathcal{U}(\mathcal{G}_\hbar)$ (where \mathcal{G}_\hbar is the Lie algebra with Lie bracket rescaled by a \hbar) to a quantum G -space $(C^\infty(M)[[\hbar]], \star)$ (satisfying also Condition (2)). As already noticed in [25], one can equivalently use the Gutt star-algebra $(C^\infty(\mathcal{G}^*)[[\hbar]], \star_G)$ as domain for the quantum momentum map.

2.3. Main results. One difficulty in the theory of quantum momentum maps is that explicit examples and formulas are hard to come by (except notably for [16]). The main result of this note consists in an explicit formula for a family of deformation quantizations (parametrized by formal G -systems) for the momentum map associated with the cotangent lift of a Lie group action on \mathbb{R}^d . Although our quantizations do not satisfy Conditions (1) and (2) of Ping Xu's original definition (Definition 8), they satisfy deformations of them, controlled by formal G -systems. Let us recall some terminology.

A smooth action φ of a Lie group G on \mathbb{R}^d determines, via cotangent lift, an Hamiltonian action $\tilde{\varphi}$ of G on the cotangent bundle $T^*\mathbb{R}^d$, which we identify with $\mathbb{R}^{2d} = \mathbb{R}_x^d \oplus \mathbb{R}_\xi^d$ endowed with the canonical symplectic form $\omega = \sum_i d\xi^i \wedge dx^i$. This action has a momentum map $J : T^*\mathbb{R}^d \rightarrow \mathcal{G}^*$, where \mathcal{G}^* is the dual of the Lie algebra \mathcal{G} of the Lie group G . It is given by

$$(2.10) \quad J(x, \xi) = -\langle \xi, X^\cdot(x) \rangle$$

where $X^\cdot : \mathcal{G} \rightarrow \text{Vect}(\mathbb{R}^d)$ is the induced infinitesimal action, i.e., $X^v(x)$ is the fundamental vector field associated with the element $v \in \mathcal{G}$. The sign in (2.10) comes from choosing the symplectic form on the cotangent bundle to be ω as above (instead of $-\omega$).

Let us state here the main theorem, which we will prove later on.

Theorem 10. *Let $a \in \mathcal{P}_\varphi^1$ be a formal G -system associated with a smooth action φ of a Lie group G on \mathbb{R}^d . The asymptotic expansion in the limit $\hbar \rightarrow 0$ of the map*

$$(2.11) \quad J^a(u)(x, \xi) = e^{-\frac{i\langle x, \xi \rangle}{\hbar}} \int_{\mathcal{G} \times \mathcal{G}^*}^{\text{formal}} u(\theta) a_{\exp(v)}(x, \xi) e^{\frac{i}{\hbar} S_{(x, \xi)}(\langle \theta, v \rangle)} \frac{dvd\theta}{(2\pi\hbar)^{\dim G}}$$

where

$$(2.12) \quad S_{(x, \xi)}(\theta, v) = \langle \xi, \varphi_{\exp(-v)}(x) \rangle - \langle \theta, v \rangle$$

is a deformation quantization of the momentum map J above from $C^\infty(\mathcal{G}^*)[[\hbar]]$, endowed with the Gutt star-product to $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ endowed with the standard star-product. The integral sign \int^{formal} means that (2.11) is identified its asymptotic expansion in \hbar in the limit $\hbar \rightarrow 0$, as prescribed by the stationary phase theorem (see remark below).

Remark 11. Analytical meaning of (2.11). The phase $S_{x,\xi}(v, \theta)$ in (2.11) should be actually understood only as a germ of function, since the exponential map \exp is only defined from a neighborhood of 0 in the Lie algebra. Therefore the phase is not defined on whole integration domain $T^*\mathcal{G}^*$, but only on a neighborhood of its zero section (which depends on the germ representative we choose). For (2.11) to make sense as an integral, one needs to throw in the integral a compactly supported cutoff function $\chi_{(x,\xi)}(\theta, v)$ whose support contains the critical point $(J(x, \xi), 0)$. With this cutoff function, the integral becomes absolutely convergent, and all the operations permitted for absolutely convergent integrals will now apply. (This observation will justify the computations we will perform later on to prove, among other things, that J^a is a unital algebra morphism.)

The problem at this point is that, if we choose another cutoff function, the value of the integral will change, since, in fact, we integrate over a different domain. To remedy this, one should consider the limit $\hbar \rightarrow 0$ after integration, which does not depend on the choice of the cutoff function, as the stationary phase Theorem guarantees (see [11, 21]). We then identify (2.11) with its asymptotic expansion in \hbar in the limit $\hbar \rightarrow 0$, which is independent of any cutoff function, leaving the integral always well defined: This is the meaning of the special integration sign \int^{formal} in (2.11).

This remark will apply to all integrals we encounter in this paper. For the sake of notational brevity, we will avoid to put the cutoff function each time, and it will be understood that we are dealing with the asymptotic expansion (a formal power series) of the resulting absolutely convergent integral. Moreover, again for the sake of notational simplicity, we will use the standard integral sign \int instead of the more correct \int^{formal} for most of the integrals coming in the remaining of this paper.

Note also that, at times, integral (2.11) also makes sense as a non-formal integral (provided one chooses an appropriate space of functions on \mathcal{G}^*) as the following example shows (we will come back to this issue at the end of this paragraph in Remark (16)):

Example 12. Consider with the action of \mathbb{R}^d on itself by translations and take a to be the trivial G -system: $a = 1$. In this case, the quantized action is $T_v^a \psi(x) = \psi(x - v)$, its lift to the cotangent bundle is $(\tilde{T}_v^a f)(x, \xi) = f(x - v, \xi)$, and the corresponding quantum momentum map is

$$J^a(u)(x, \xi) = \int_{\mathbb{R}^{2d}} u(\bar{\theta}) e^{\frac{i}{\hbar} \langle \bar{v}, -\xi - \bar{\theta} \rangle} \frac{d\bar{v} d\bar{\theta}}{(2\pi\hbar)^d} = u(-\xi).$$

Here it is easy to see that J^a is a unital algebra morphism: the constant function 1 on $(\mathbb{R}^d)^*$ is sent to the constant function 1 on $T^*\mathbb{R}^d$; the product of two functions hk on $(\mathbb{R}^d)^*$ (which corresponds to the Gutt star-product when the Lie group we start with is abelian) is sent to $J^a(hk)(x, \xi) = h(-\xi)k(-\xi)$, which is the standard product of two functions on $T^*\mathbb{R}^d$ depending only on ξ (it comes from the asymptotic expansion of \star_{st} , see [11, 21]).

The quantizations J^a we propose in Theorem 10 do not satisfy Conditions (1) and (2) of Definition 8 but rather deformations of them controlled by G -systems. Namely, the standard product defined by the composition of pseudo-differential operators as in (2.6) is in general not G -equivariant for cotangent lift actions (unless the action on \mathbb{R}^d we start with is linear). Thus, $(C^\infty(T^*\mathbb{R}^d)[[\hbar]], \star_{st})$ is in general

not a quantum G -space in the sense of Definition (8), and Condition (1) is not satisfied. However, we have the following:

Proposition 13. *Let a be a formal G -system and let \tilde{T}^a be the induced action on $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ as in (2.1). The standard product is G -equivariant for this action:*

$$(\tilde{T}_g^a f) \star_{st} (\tilde{T}_g^a g) = \tilde{T}_g^a (f \star_{st} g),$$

for all $g \in G$ and $f, g \in C^\infty(T^*\mathbb{R}^d)[[\hbar]]$.

Proof. Using the definition of \tilde{T}^a in (2.1) and that of the standard product, we have that

$$\begin{aligned} \text{Op}((\tilde{T}_g^a f) \star_{st} (\tilde{T}_g^a g)) &= \text{Op}(\tilde{T}_g^a f) \circ \text{Op}(\tilde{T}_g^a g) \\ &= \tilde{T}_g^a \circ \text{Op}(f) \circ \tilde{T}_{g^{-1}}^a \circ \tilde{T}_g^a \circ \text{Op}(g) \circ \tilde{T}_g^a \\ &= \tilde{T}_g^a \text{Op}(f \star_{st} g) \tilde{T}_{g^{-1}}^a \\ &= \text{Op}(\tilde{T}_g^a (f \star_{st} g)), \end{aligned}$$

which proves our claim, since Op is injective. \square

Thus $(C^\infty(T^*\mathbb{R}^d)[[\hbar]], \star_{st})$ is a kind of quantum G -space but for the deformed action $\tilde{T}^a = \tilde{\varphi}^* + \mathcal{O}(\hbar)$ (note that corrections in \hbar are present even in the case a is the trivial G -system $a = 1$). Condition (2) has also a deformed analog, which we will be able to prove only later on though:

Theorem 14. *For all $v \in \mathcal{G}$, seen as a linear function on \mathcal{G}^* , we have that*

$$(2.13) \quad J^a(v) = J^*v + \frac{\hbar}{i}(D_e a)v, \quad t_v^a = \text{Op}\left(\frac{i}{\hbar}J^a(v)\right), \quad \frac{i}{\hbar}[J^a(v), f] = \tilde{t}_v^a(f),$$

where $t_v^a = (D_e T^a)v$, $\tilde{t}_v^a = (D_e \tilde{T}^a)v$, and D_e is the derivative w.r.t. the group variable evaluated at the group unit e .

Since $\tilde{t}_v^a f = \tilde{X}^v f + \mathcal{O}(\hbar)$, where \tilde{X}^v is the fundamental vector field for the cotangent lift action (associated with $v \in \mathcal{G}$), we have that the last identity in (2.13) is a deformation of Ping Xu's second condition in Definition 8.

Remark 15. For the trivial G -system $a = 1$, we have, by (2.13), that J^a coincides with the classical momentum map on linear elements and, thus, that $t_v^a = X^v$, which agrees with the fact that, in this case, the induced action is the pullback action $T_g^a \psi(x) = \psi(\varphi_g^{-1}(x))$. However, the induced action \tilde{T}^a on $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$ defined by (2.1) does not coincide in general with the action by cotangent lift pullbacks, even if $a = 1$. Therefore its derivative \tilde{t}^a is also in this case a deformation of the action by cotangent lifts of fundamental vector fields.

There is a case though when \tilde{T}^a coincides with the action by cotangent lift pullbacks: namely, when the action φ on \mathbb{R}^d we start with is linear or affine as in Example 12. (One sees this directly from (2.1), since, in the linear case, there is no corrections in \hbar .) This implies that $\tilde{t}_v^a = \tilde{X}^v$ and that Ping Xu's second condition is exactly satisfied. Therefore, for linear or affine actions and the trivial G -system, J^a is a quantum momentum map in the sense of [25]. Actually, our formula provides an explicit formula to [25, Example 6.5], where the quantum momentum map was computed only on linear elements.

Let us close this section by a remark on the geometrical meaning of the oscillatory integral (2.11) defining our quantization family.

Remark 16. Geometrical meaning of (2.10). As shown in [8], a Poisson map from B to A integrates to a symplectic micromorphism from T^*A to T^*B , which is a special lagrangian submanifold germ of $\overline{T^*A} \times T^*B$. These symplectic micromorphisms always possess a global generating function (see [7]), which allows us to quantize them (i.e. to associate with them formal operators from $C^*(A)[[\hbar]]$ to $C^*(B)[[\hbar]]$, see [9]) using Fourier integral operator techniques. Formula (2.11) can be seen as such a quantization, where the symplectic micromorphism involved is the one that integrates the classical momentum map J , regarded as a Poisson map.

When the Poisson map is complete (i.e. when it pulls back complete hamiltonian vector fields to complete hamiltonian vector fields), the integrating symplectic micromorphism can be extended to a global lagrangian submanifold, namely a symplectic comorphism (see [5]). In this case, one can expect to obtain bounded operators from some functional space on A to some other functional space on B as quantization, instead of their formal asymptotic expansions. Here, this is reflected by the fact that the exponential map is defined on the whole Lie algebra only for certain types of Lie groups (e.g. for the nilpotent ones). For those, the phase becomes a true function defined on the whole integration domain. It would be interesting (although analytically challenging) to see if our construction can go beyond the formal and asymptotic case for nilpotent groups. However, it is not completely clear to us what are the right functional spaces to be considered as replacements for the formal spaces $C^\infty(\mathcal{G}^*)[[\hbar]]$ and $C^\infty(T^*\mathbb{R}^d)[[\hbar]]$.

3. PROOFS OF THE MAIN RESULTS

In this section, we give proofs of Theorem 10 and Theorem 14. One of the main tools will be an asymptotic expansion of (2.11) in the limit $\hbar \rightarrow 0$, using the standard Feynman graphical methods. We start off by recalling some basic facts about the Feynman calculus.

3.1. Feynman asymptotic expansions. Let us start with a reminder about asymptotic expansions of oscillatory integrals in terms of Feynman graphs (we refer the reader to [4] for more details). Consider the integral

$$(3.1) \quad I(\hbar) = e^{-\frac{i}{\hbar}S(c)} \int_{\mathbb{R}^d}^{\text{formal}} g_1(z) \cdots g_n(z) e^{\frac{i}{\hbar}S(z)} \frac{dz}{(2\pi\hbar)^{\frac{d}{2}}},$$

where S is a smooth function on \mathbb{R}^d with a unique non-degenerate critical point c , g_1, \dots, g_n are smooth functions on \mathbb{R}^d , and (3.1) is to be understood as an asymptotic expansion in the limit $\hbar \rightarrow 0$, yielding a formal power series in \hbar , in the sense of Remark 11.

Feynman's Theorem gives the asymptotic expansion (3.1) as a sum over certain graphs: namely,

$$(3.2) \quad I(\hbar) = \frac{e^{\frac{i\pi}{4} \text{sign } B}}{\sqrt{\det |B|}} \left(\sum_{\Gamma \in G_{3 \geq}(n)} \frac{(i\hbar)^{|E_\Gamma| - |V_\Gamma^{\text{int}}|}}{|\text{Aut}(\Gamma)|} F_\Gamma(S; g_1, \dots, g_n) \right),$$

where $B = D^2S(c)$, Γ is a Feynman graph with n external vertices, $|E_\Gamma|$ is its number of edges, $|V_\Gamma^{\text{int}}|$ is its number of internal vertices, F_Γ is the corresponding

Feynman amplitude, and $|\text{Aut } \Gamma|$ is the number of symmetries of Γ . Let us now explain what we mean by Feynman's graphs and amplitudes.

A **Feynman graph** is a (non-oriented) graph whose vertex set V_Γ is partitioned into two disjoint sets $V_\Gamma = V_\Gamma^{\text{ext}} \sqcup V_\Gamma^{\text{int}}$: the set of **external vertices** V_Γ^{ext} , which we will represent on an imaginary line, as in Table 1, and the set of **internal vertices** V_Γ^{int} , which we will represent above this imaginary line. Multi-edges and loops are allowed, but each internal vertex must have valence greater or equal to 3. We denote by $G_{3\geq}(n)$ the set of isomorphism classes of Feynman graphs with n external vertices. We now turn to Feynman's amplitudes.

Let S and g_1, \dots, g_n be smooth functions on \mathbb{R}^d as above. Given a Feynman graph $\Gamma \in G_{3\geq}(n)$ with k internal vertices, the corresponding **Feynman amplitude** $F_\Gamma(S; g_1, \dots, g_n)$ is a product of k partial derivatives of S (represented by the internal vertices) and the partial derivatives of the g_i 's (represented by the external vertices) all of which are evaluated at the critical point c and contracted using the tensor B^{-1} (i.e. the inverse of the Hessian matrix of S evaluated at c). The Feynman graph records which partial derivatives are involved and how contractions of these partial derivatives are to be done. We can summarize the procedure as follows:

- (1) Label the two extremities of each edge with a coordinate index in $\{1, \dots, n\}$
- (2) An internal vertex with l incoming edges labelled with i_1, \dots, i_l will produce a factor $\frac{\partial^l S}{\partial z^{i_1} \dots \partial z^{i_l}}(c)$ in the amplitude
- (3) An external vertex $j \in V_\Gamma^{\text{int}}$ with l incoming edges labelled with i_1, \dots, i_l will produce a factor $\frac{\partial^l g_j}{\partial z^{i_1} \dots \partial z^{i_l}}(c)$ in the amplitude
- (4) Each edge whose extremities are labelled with, say, i and j (such a labelled edge is also called a **propagator**) will produce a factor $(B^{-1})^{ij}$ in the amplitude

The resulting terms should be summed up using the Einstein summation convention. Here is a table representing a few Feynman graphs and their amplitudes to illustrate the process:



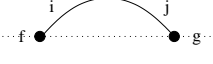
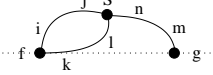
	\rightsquigarrow	$f(c)g(c)$
	\rightsquigarrow	$(B^{-1})^{ij} \frac{\partial^2 f}{\partial z^i \partial z^j}(c) g(c)$
	\rightsquigarrow	$(B^{-1})^{ij} \frac{\partial f}{\partial z^i}(c) \frac{\partial g}{\partial z^j}(c)$
	\rightsquigarrow	$(B^{-1})^{ij} (B^{-1})^{kl} (B^{-1})^{nm} \frac{\partial^3 S(c)}{\partial z^j \partial z^l \partial z^n} \frac{\partial f(c)}{\partial z^i \partial z^k} \frac{\partial g(c)}{\partial z^m}$

TABLE 1. On the left some Feynman graphs Γ in $G_{3\geq}(2)$ and on the right their corresponding amplitudes $F_\Gamma(S; f, g)$, written using the Einstein summation convention. We decorated the graphs with labels to make the correspondence more transparent.

3.2. Explicit asymptotic expansion for J^a . We now want to use Feynman's theorem (3.2) to obtain an explicit formula for our family J^a of momentum map quantizations (2.11) in Theorem 10.

The first order of business is to check that the phase in this integral has a unique non-degenerate critical point and then to compute the determinant, the signature and the inverse of its Hessian matrix at this point. Once done, we can use (3.2) in a straightforward manner.

Lemma 17. *Consider the phase of integral (2.11):*

$$S_{x,\xi}(v, \theta) = \langle \xi, \varphi_{\exp(-v)}x \rangle - \langle \theta, v \rangle.$$

Then, for each $(x, \xi) \in T^\mathbb{R}^d$, the phase has a unique critical point w.r.t. to the $v\theta$ -variables; namely,*

$$(v_0 = 0, \quad \theta_0 = J(x, \xi)),$$

where $J(x, \xi) = -\langle \xi, X^v(x) \rangle$ is the value of the classical momentum map at (x, ξ) . Moreover, at this critical point, we have that

$$\det(D^2 S_{x,\xi}(v_0, \theta_0)) = 1 \quad \text{and} \quad \text{sign}(D^2 S_{x,\xi}(v_0, \theta_0)) = 0,$$

and the inverse of the Hessian matrix B is

$$B^{-1} = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & -\langle \xi, D^2 \varphi.(x) \rangle \end{pmatrix}.$$

Proof. We get the unique critical point from a direct computation and the fact that

$$\frac{d}{dv} \varphi_{\exp(-v)}(x)|_{v=0} = -X^v(x).$$

The Hessian matrix of the phase at that point is

$$B = \begin{pmatrix} \langle \xi, D^2 \varphi.(x) \rangle & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

from which we get the form of its inverse as well as the fact that the absolute value of its determinant is always 1 for all values (x, ξ) . We now prove that the signature of B (i.e. the number of positive eigenvalues minus the number of negative eigenvalues) is always equal to zero. First of all, observe that, at $\xi = 0$, the signature of

$$D^2 S_{x,\xi}(v_0, \theta_0)|_{\xi=0} = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix},$$

is zero. Suppose that there exists a $\bar{\xi}$ such that the signature of $D_{x,\bar{\xi}}^2 S(v_0, \theta_0)$ is non-zero, and consider the function

$$f : t \mapsto |\det(D_{x,t\bar{\xi}}^2 S(v_0, \theta_0))|,$$

which is identically equals to 1 for all values of $t \in [0, 1]$. However, the signature of $D_{x,t\bar{\xi}}^2 S(v_0, \theta_0)$ is zero at $t = 0$ and, by assumption, different from zero at $t = 1$. This means that the sign of at least one of the eigenvalues λ_t of $D_{x,t\bar{\xi}}^2 S(v_0, \theta_0)$ must have changed between $t = 0$ and $t = 1$. This implies that there is a value $t_0 \in [0, 1]$ for which $\lambda_{t_0} = 0$, and, consequently, that $f(t_0) = 0$, which contradicts the fact that this function is identically 1. \square

The previous Lemma tells us that we can use the Feynman expansion (3.2) for our quantization family (2.11), which yields

$$\begin{aligned}
 (J^a u)(x, \xi) &= \sum_{\Gamma \in G_{3 \geq}(2)} \frac{(i\hbar)^{|E_\Gamma| - |V_\Gamma^{\text{int}}|}}{|\text{Aut } \Gamma|} F_\Gamma(S; u, a) \\
 (3.3) \qquad &= \sum_{\Gamma \in G_{3 \geq}(2)} \sum_{l=0}^{\infty} \frac{(-i)^l (i\hbar)^{|E_\Gamma| - |V_\Gamma^{\text{int}}| + l}}{|\text{Aut } \Gamma|} F_\Gamma(S; u, P^l)
 \end{aligned}$$

where u and the P^l 's are seen as functions of $z = (v, \theta)$ with the particularity that u depends only on θ and P^l depends only on v . Because of this particularity and the special form of B^{-1} many Feynman graphs will have zero amplitude.

We now want to find out how the non-vanishing Feynman graphs look like. We start with a lemma whose proof can be read off directly from the form of the phase S and the form of B^{-1} in Lemma 17:

Lemma 18. *The only non-vanishing propagators are of two kinds:*

(1) *edges for which one extremity is labelled by v^i while the other extremity is labelled by θ_i ($i = 1, \dots, \dim G$). The corresponding term in the amplitude is (-1) .*

(2) *edges for which one extremity is labelled by θ_i while the other extremity is labelled by θ_j ($i, j = 1, \dots, \dim G$). The corresponding term in the amplitude is $-\sum_k \xi_k (D_e^2 \varphi^k)_{ij}$.*

The only non-vanishing internal vertices of valence k are those whose incoming edges have labels v_{i_1}, \dots, v_{i_k} (i.e. none of the labels is taken in the θ -coordinates).

In Table 2, we depict the non-vanishing propagators and vertices entering in the Feynman expansion of J^a .

	\rightsquigarrow	$(B^{-1})^{v\theta} = -1$
	\rightsquigarrow	$(B^{-1})^{\theta_i \theta_j} = -\xi_k (D_e^2 \varphi^k)_{ij}$
	\rightsquigarrow	$\frac{\partial^l S_{x, \xi}(v_0, \theta_0)}{\partial v^{i_1} \dots \partial v^{i_l}} = (-1)^l \xi_k (D_e^l \varphi^k)_{i_1 \dots i_l}$
	\rightsquigarrow	$\frac{\partial^l a_e(x, \xi)}{\partial v^{i_1} \dots \partial v^{i_l}}$
	\rightsquigarrow	$\frac{\partial^l u(J(x, \xi))}{\partial \theta_{i_1} \dots \partial \theta_{i_l}}$

TABLE 2. The non-vanishing propagators and vertices entering in the Feynman expansion of J^a .

Corollary 19. *The only Feynman graphs $\Gamma \in G_{\geq 3}(2)$ whose amplitudes $F_{\Gamma}(u, a)$ do not vanish for all $u \in C^{\infty}(\mathcal{G}^*)$ and formal G -system a are of the form depicted in Figure 3.1.*

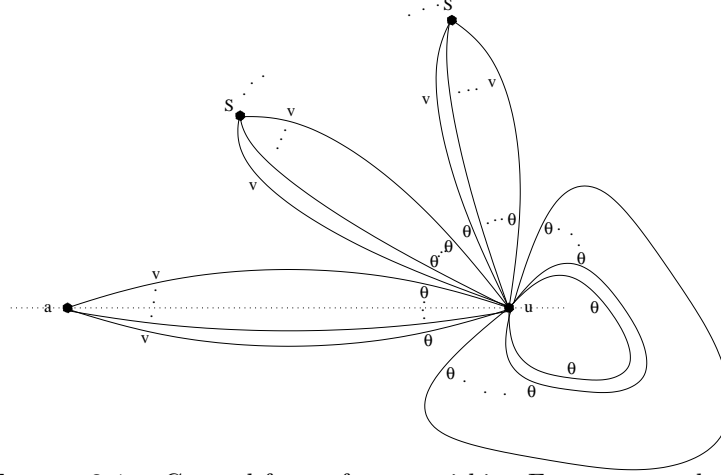


FIGURE 3.1. General form of non-vanishing Feynman graphs in the expansion of J^a .

3.3. Proof of Theorem 10. We are now in measure to prove Theorem 10. We split its proof into a series of lemmas. The first one shows that J^a has the right first term to be candidate for a deformation quantization of the classical momentum map:

Lemma 20. *The first term of the expansion (3.3) is*

$$(3.4) \quad J^a u = J^* u + \mathcal{O}(\hbar),$$

where J^* is the pullback of the classical momentum map (2.10).

Proof. The Feynman graph with two external vertices, no internal vertex, and no edge is the first term of the expansion. The amplitude corresponding to this term is

$$F_{\Gamma}(a, u) = u(\theta_0) a_{\exp(v_0)}(x, \xi) = (J^* u)(x, \xi),$$

since, by Lemma 17, the unique critical point is given by $\theta_0 = J(x, \xi)$ and $v_0 = 0$, and the formal G -system evaluated at the unit is identically 1 (i.e. $P_e^0(x) = 1$ and $P_e^n(x, \xi) = 0$ for $n \geq 1$). \square

Let us show now the unitality part of Theorem 10:

Lemma 21. *We have that $J^a(1) = 1$.*

Proof. Using the Feynman expansion (3.3), we have that the only graph $\Gamma \in G_{\geq 3}(2)$ such that $F_{\Gamma}(a, 1) \neq 0$ is the one with no edge (any other non-vanishing graph as in Figure 3.1 involves derivatives of the constant function 1). \square

To complete the proof of Theorem 10, that is, to show that J^a is an algebra morphism, we need to wait until the next section. However, we are ready for the proof of Theorem 14.

In this section, we will prove that the map

$$J^a : (C^\infty(\mathcal{G}^*)[[\hbar]], \star_G) \longrightarrow (C^\infty(T^*\mathbb{R}^d)[[\hbar]], \star_{st})$$

defined in Theorem 10 is an algebra homomorphism, i.e., it is a quantization of the classical momentum map of a cotangent lift action J in (2.10) (we already know from last section that $J^a(1) = 1$ and that $J^a u = J^* u + \mathcal{O}(\hbar)$). This will complete the proof of Theorem 10.

At last, we prove that J^a is an algebra morphism. Most of the following computations are formal but can be made rigorous by throwing in suitable compactly supported cutoff functions in the integrals as explained in Remark 11. For convenience, we define for $v \in \mathcal{G}$ the following function on \mathcal{G}^* :

$$(3.5) \quad e_v(\theta) = e^{\frac{i}{\hbar} \langle v, \theta \rangle},$$

whose asymptotic Fourier transform is the translated delta function,

$$\mathcal{F}_\hbar(e_v)(w) = \delta(w - v),$$

where the asymptotic Fourier transform of a distribution on \mathbb{R}^n is defined by

$$\mathcal{F}_\hbar f(\xi) = \int f(x) e^{-\frac{i}{\hbar} \langle \xi, x \rangle} \frac{dx}{(2\pi\hbar)^{\frac{n}{2}}}.$$

The following lemma is a standard property of the Gutt star-product, which we reprove here:

Lemma 22. *For all $v, w \in \mathcal{G}$, we have*

$$(3.6) \quad e_v *_G e_w = e_{BCH(v, w)}$$

Proof. Let $v_0, w_0 \in \mathcal{G}$. Then we have

$$\begin{aligned} (e_{v_0} *_G e_{w_0})(\theta) &= \int_{\mathcal{G} \times \mathcal{G}^*} \mathcal{F}_\hbar(e_{v_0})(v) \mathcal{F}_\hbar(e_{w_0})(w) e^{\frac{i}{\hbar} \langle \theta, BCH(v, w) \rangle} \frac{dv dw}{(2\pi\hbar)^{\dim G}} \\ &= \int_{\mathcal{G} \times \mathcal{G}^*} \delta(v - v_0) \delta(w - w_0) e^{\frac{i}{\hbar} \langle \theta, BCH(v, w) \rangle} \frac{dv dw}{(2\pi\hbar)^{\dim G}} \\ &= e^{\frac{i}{\hbar} \langle \theta, BCH(v_0, w_0) \rangle} = e_{BCH(v_0, w_0)}(\theta). \end{aligned}$$

□

Lemma 23. *For any $v \in \mathcal{G}$, seen as a linear function on \mathcal{G}^* , the following identity holds true:*

$$(3.7) \quad Op(J^a(e_v)) = T_{\exp(v)}^a.$$

Proof. First we note that

$$\begin{aligned} (J^a e_v)(\xi, x) &= e^{-\frac{i}{\hbar} \langle \xi, x \rangle} \int_{\mathcal{G}} \delta(w - v) a_{\exp(w)}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{\exp(-w)}(x) \rangle} \frac{dw}{(2\pi\hbar)^{\frac{\dim G}{2}}} \\ &= e^{-\frac{i}{\hbar} \langle \xi, x \rangle} a_{\exp(v)}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{\exp(-v)}(x) \rangle}, \end{aligned}$$

which implies that

$$\begin{aligned}
 \text{Op}(J^a(e_v))\psi(x) &= \int_{\mathbb{R}^{2d}} \psi(\bar{x}) J^a(e_v)(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, x - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^d} \\
 (3.8) \qquad \qquad \qquad &= \int_{\mathbb{R}^{2d}} \psi(\bar{x}) a_{\exp(v)}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{\exp(-v)}(x) - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^d}.
 \end{aligned}$$

which proves the lemma. \square

Lemma 24. *For every $v, w \in \mathcal{G}$ we have*

$$(3.9) \qquad J^a(e_v \star_G e_w) = J^a(e_v) \star_{st} J^a(e_w).$$

Proof. We check this identity at the level of the corresponding pseudo-differential operators. Then, from Lemma 22 and 23, we obtain

$$\text{Op}(J^a(e_v \star_G e_w)) = \text{Op}(J^a(e_{\text{BCH}(v,w)})) = T_{\exp}^a(\text{BCH}(v,w)).$$

On the other hand

$$\text{Op}(J^a(e_v) \star_{st} J^a(e_w)) = \text{Op}(J^a(e_v)) \circ \text{Op}(J^a(e_w)) = T_{\exp(v)}^a \circ T_{\exp(w)}^a,$$

and we can conclude the proof by invoking the injectivity of Op and the fact that

$$T_{\exp}^a(\text{BCH}(v,w)) = T_{\exp(v)}^a \circ T_{\exp(w)}^a,$$

since the operators $\{T_g^a\}_{g \in G}$ define a representation of the Lie group G . \square

Then we can conclude the proof of Theorem 10:

Proposition 25. *The map J^a is an algebra morphism.*

Proof. Suppose $\dim \mathcal{G} = n$. Let $f, g \in C^\infty(\mathcal{G}^*)$ and let us consider their Fourier decomposition:

$$f(\theta) = \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) e_v(\theta) \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \quad \text{and} \quad g(\theta) = \int_{\mathcal{G}} \mathcal{F}_\hbar g(w) e_w(\theta) \frac{dw}{(2\pi\hbar)^{\frac{n}{2}}}.$$

Then we compute

$$\begin{aligned}
 J^a(f \star_G g) &= J^a \left(\left(\int_{\mathcal{G}} \mathcal{F}_\hbar f(v) e_v(\theta) \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \right) \star_G \left(\int_{\mathcal{G}} \mathcal{F}_\hbar g(w) e_w(\theta) \frac{dw}{(2\pi\hbar)^{\frac{n}{2}}} \right) \right), \\
 &= \int_{\mathcal{G} \times \mathcal{G}} J^a(e_v \star_G e_w) \mathcal{F}_\hbar f(v) \mathcal{F}_\hbar g(w) \frac{dv dw}{(2\pi\hbar)^n} \\
 &= \int_{\mathcal{G} \times \mathcal{G}} (J^a(e_v) \star_{st} J^a(e_w)) \mathcal{F}_\hbar f(v) \mathcal{F}_\hbar g(w) \frac{dv dw}{(2\pi\hbar)^n} \\
 &= J^a \left(\int_{\mathcal{G}} \mathcal{F}_\hbar f(v) e_v \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \right) \star_{st} J^a \left(\int_{\mathcal{G}} \mathcal{F}_\hbar g(w) e_w \frac{dw}{(2\pi\hbar)^{\frac{n}{2}}} \right) \\
 &= J^a(f) \star_{st} J^a(g),
 \end{aligned}$$

which concludes the proof. \square

3.4. Proof of Theorem 14. Let us start with the first identity in Theorem 14.

Proposition 26. *For $v \in \mathcal{G}$, seen as a linear function on \mathcal{G}^* , the following identity holds true*

$$(3.10) \quad J^a(v) = J^*v + \frac{\hbar}{i}(D_e a)v.$$

Proof. First we observe that there is no graph Γ with internal vertex such that $B_\Gamma(a, v) \neq 0$: Suppose that Γ has an internal vertex and that $B_\Gamma(a, v) \neq 0$. Then one and only one of the edges stemming out of this vertex must land on the external vertex labelled with v (if none is landing on v , we end up with a zero propagator, since this internal vertex has (at least) two more edges decorated by v), and if more than one edge is landing on v , then we differentiate twice a linear function.

Also notice that a Feynman graph such that $B_\Gamma(a, v) \neq 0$ can not have any self loop based on v , because any of these loops would involve at least a factor of the form $\frac{\partial l_v}{\partial v^i}$ or a factor of the form $\frac{\partial^2 l_v}{\partial \theta_i \partial \theta_j}$ in the corresponding amplitude. Since l_v is a linear function depending on θ only, both factors would yield a zero amplitude. Loops based on the external vertex decorated by a will also yield a zero amplitude, since every non-vanishing propagator involve at least one derivative in the θ -direction and a depends on the variable v only.

The only remaining possibilities are the graph Γ_0 with no internal vertex nor edge and the graph Γ_1 formed by the two external vertices decorated by a and v respectively and a single edge with label v^i on the a extremity and with label θ_i on the v extremity (multi-edges would yield multiple derivations of the linear function l_v , and hence yield zero). (We could have seen all this immediately by direct inspection of Figure 3.1.)

The amplitudes of these two graphs correspond to the two terms in (3.10). \square

Let us prove the second identity of Theorem 14.

Proposition 27. *The following identity holds true:*

$$(3.11) \quad t_v^a = Op\left(\frac{i}{\hbar}J^a(v)\right),$$

where $v \in \mathcal{G}$ is regarded as a linear function on \mathcal{G}^* .

Proof. Recall that, by definition, we have

$$t_v^a = (D_e T^a)(v) = \frac{d}{dt} T_{\exp(tv)}^a|_{t=0}, \quad v \in \mathcal{G}.$$

Then, interchanging derivation and integration and using (3.11), we obtain

$$\begin{aligned} t_v^a &= \int_{T^*\mathbb{R}^d} \psi(\bar{x}) \frac{d}{dt} \left(a_{\exp(tv)}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{\exp(-tv)}(x) \rangle} \right) \Big|_{t=0} e^{-\frac{i}{\hbar} \langle \bar{\xi}, \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}}, \\ &= \int_{T^*\mathbb{R}^d} \psi(\bar{x}) (D_e a(x, \bar{\xi})(v) - \langle \bar{\xi}, X^v(x) \rangle) e^{\frac{i}{\hbar} \langle \bar{\xi}, x \rangle} e^{-\frac{i}{\hbar} \langle \bar{\xi}, \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}}, \\ &= \int_{T^*\mathbb{R}^d} \psi(\bar{x}) \left(\frac{i}{\hbar} J^a(v) \right) e^{\frac{i}{\hbar} \langle \bar{\xi}, x - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}}, \end{aligned}$$

which concludes the proof. \square

At last, we are ready for the proof of the last identity of Theorem 14, which corresponds to a deformation of Ping Xu's second condition for quantum momentum maps:

Proposition 28. *We have that*

$$\frac{i}{\hbar}[J^a(v), f] = \tilde{t}_v^a(f)$$

for all $v \in \mathcal{G}$ and $f \in C^\infty(T^*\mathbb{R}^d)[[\hbar]]$.

Proof. Consider Equation (2.1) evaluated at $g = \exp(tv)$ with $v \in \mathcal{G}$: i.e.

$$T_{\exp(tv)}^a \text{Op}(f) T_{\exp(-tv)}^a = \text{Op}(\tilde{T}_{\exp(tv)}^a f).$$

Differentiating this last equation w.r.t. the variable t at $t = 0$, we obtain

$$[t_v^a, \text{Op}(f)] = \text{Op}(\tilde{t}_v^a f)$$

where $\tilde{t}_v^a = (D_e \tilde{T}^a)v$ for $v \in \mathcal{G}$. Then, by Proposition (27), we have that

$$[\text{Op}(\frac{i}{\hbar} J^a(v)), \text{Op}(f)] = \text{Op}(\tilde{t}_v^a f)$$

and finally that

$$\text{Op}\left(\frac{i}{\hbar}[J^a(v), f]\right) = \text{Op}(\tilde{t}_v^a f),$$

which concludes the proof by injectivity of Op . \square

4. INVARIANT HAMILTONIANS

In this section, we consider certain Hamiltonians invariant w.r.t. the cotangent lift of an action of a Lie group G on \mathbb{R}^d . In the classical case, invariant Hamiltonians can be obtained as images of invariant functions in $C^\infty(\mathcal{G}^*)^G$ by the classical comomentum map (i.e. the pullback of J defined in (2.10)). However, the quantum Hamiltonians resulting from the quantization of these invariant classical Hamiltonians are in general no longer invariant w.r.t. to the quantized action: anomalies appear. We show here how to use G -systems and their associated quantizations J^a defined in (2.11) to obtain classical invariant Hamiltonians that are still invariant upon quantization w.r.t. to the quantization T^a (associated with the same G -system a) of the action. In other words, we explain how to use G -systems to obtain both quantum symmetries and invariant Hamiltonians with no anomalies upon quantization.

Let us start by recalling the classical case.

4.1. Classical case. Let φ be a smooth action of a Lie group G on \mathbb{R}^d , and consider the corresponding hamiltonian action $\tilde{\varphi}$ on the cotangent bundle $T^*\mathbb{R}^d$ given by cotangent lift with momentum map J given by (2.10). We will denote by $Ad_g^\#$ the coadjoint action of G on \mathcal{G}^* . It induces an action on $C^\infty(\mathcal{G}^*)$ by pullbacks, which we will still denote the same way:

$$(Ad_g^\# f)(\alpha) = f(Ad_{g^{-1}}^\# \alpha)$$

for all $f \in C^\infty(\mathcal{G}^*)$ and $\alpha \in \mathcal{G}^*$. Equivariance of the momentum map implies equivariance of its pullback, the comomentum map:

Lemma 29. *For every $f \in C^\infty(\mathcal{G}^*)$, the following identity holds true:*

$$J^*(Ad_g^\# f) = \tilde{\varphi}_{g^{-1}}^* J^*(f).$$

Definition 30. We denote by $C^\infty(\mathcal{G}^*)^G$ the space of $Ad^\#$ -invariant (i.e. $Ad_g^\# f = f$ for all $g \in G$). Observe that this space coincides with the center Z of the Lie algebra $(C^\infty(\mathcal{G}^*), \{, \})$.

From the lemma above, we have that

Corollary 31. *Let $f \in C^\infty(\mathcal{G}^*)^G$ be an $Ad^\#$ -invariant function. Then, its image by J^* is invariant w.r.t. the cotangent lift action:*

$$(4.1) \quad \tilde{\varphi}_g^* J^* f = J^* f.$$

for all $g \in G$

We will denote the invariant functions on $T^*\mathbb{R}^d$ as in Corollary 31 by

$$H_f := J^* f, \quad f \in C^\infty(\mathcal{G}^*)^G,$$

and call them simply **invariant Hamiltonians**.

4.2. Quantum case. Given an invariant Hamiltonian H_f as in the last paragraph, its quantization (here we take the standard quantization for simplicity), i.e. the pseudo-differential operator

$$\hat{H}_f := \text{Op}(H_f)$$

is in general not a quantum invariant Hamiltonian for the trivial quantization of the symmetries, since

$$(4.2) \quad T_g \hat{H}_f T_{g^{-1}} = \varphi_g^* \hat{H}_f \varphi_{g^{-1}} = \hat{H}_f + \mathcal{O}(\hbar),$$

as exemplified by Egorov's theorem (see [21] for instance). The corrections in \hbar in (4.2) preventing \hat{H}_f to be an invariant quantum hamiltonian (i.e. $T_g \hat{H}_f T_{g^{-1}} = \hat{H}_f$) are called **anomalies**.

Given a formal G -system a , we deform both an invariant Hamiltonian H_f by using the corresponding quantization of the momentum map

$$H_f^a := J^a(f) = H_f + \mathcal{O}(\hbar),$$

and the quantum symmetries by using T^a instead of T :

$$T_g^a A T_{g^{-1}}^a = \varphi_g^* A \varphi_{g^{-1}}^* + \mathcal{O}(\hbar),$$

where A is a pseudo-differential operator. We will show that, in this case, H_f^a is still invariant as a classical Hamiltonian (i.e w.r.t. the cotangent lift action), and, moreover, that the quantization can be performed without anomalies: i.e.

$$T_g^a \hat{H}_f^a T_{g^{-1}}^a = \hat{H}_f^a,$$

where the quantum Hamiltonian is now $\hat{H}_f^a := \text{Op}(H_f^a)$. Let us start by a (rather trivial) example:

Example 32. Consider the affine action $\varphi_v(x) = x + v$ of \mathbb{R}^d on itself as in Example 12. Then the trivial G -system $a = 1$ gives $J^a f(\xi) = f(-\xi)$, which is obviously invariant by cotangent lifts of the action. The center Z is the whole space of functions on $(\mathbb{R}^d)^*$, since the Lie group is abelian, and the classical invariant Hamiltonians is the space of function $J^a(Z)$ on the cotangent bundle depending on the impulsions only. Clearly, quantization happens in this case without anomalies. However, this

situation is quite degenerate, since, here, J^a coincides with the comomentum map J^* . Observe that taking $f(\xi) = \xi^2$ in the center, $J^a f(\xi) = \xi^2$ corresponds to the Hamiltonian of the free particle, whose quantization H_f^a is the Laplace operator, which is invariant by quantization of the translations. Here, the trivial G -system gives back the usual story.

The main result of this paragraph is the following

Theorem 33. *Let a be a formal G -system, then*

$$(4.3) \quad T_g^a \text{Op}(J^a(f)) T_{g^{-1}}^a = \text{Op}(J^a(Ad_g^\sharp f)),$$

for all $f \in C^\infty(\mathcal{G}^*)$.

This result has the following consequence:

Corollary 34. *Given an Ad^\sharp -invariant function $f \in C^\infty(\mathcal{G}^*)^G$, then the corresponding quantum Hamiltonian $\hat{H}_f^a = \text{Op}(J^a(f))$ is invariant w.r.t. to the quantum symmetries T^a , i.e.*

$$(4.4) \quad T_g^a \hat{H}_f T_{g^{-1}}^a = \hat{H}_f$$

for all $g \in G$.

Remark 35. From (4.4) and (2.1), we observe that H_f^a is also invariant as a classical Hamiltonian. In fact:

$$\text{Op}(J^a(f)) = T_g^a \text{Op}(J^a(f)) T_{g^{-1}}^a = \text{Op}(\tilde{\varphi}_{g^{-1}}^* J^a(f) + \mathcal{O}(\hbar)),$$

for all $f \in C^\infty(\mathcal{G}^*)$ and $g \in G$, from which we get $\tilde{\varphi}_{g^{-1}}^* J^a(f) = J^a(f)$ for all $g \in G$.

The remaining of this section is devoted to the proof of Theorem 33.

4.3. Proof of the Theorem 33. Throughout this section, we suppose that $\dim \mathcal{G} = n$. We start with the following

Lemma 36. *Let $f \in C^\infty(\mathcal{G}^*)$. Then, for all $g \in G$*

$$(4.5) \quad \mathcal{F}_\hbar(Ad_g^\sharp f)(v) = |\det Ad_g^\sharp| \mathcal{F}_\hbar(f)(Ad_{g^{-1}}(v))$$

where \mathcal{F}_\hbar denotes the asymptotic Fourier transform.

Proof. This is a direct computation:

$$\begin{aligned} \mathcal{F}_\hbar(Ad_g^\sharp f)(v) &= \int_{\mathcal{G}^*} (Ad_g^\sharp f)(\theta) e^{-\frac{i}{\hbar} \langle \theta, v \rangle} \frac{d\theta}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= \int_{\mathcal{G}^*} f(Ad_{g^{-1}}^\sharp \theta) e^{-\frac{i}{\hbar} \langle \theta, v \rangle} \frac{d\theta}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= \int_{\mathcal{G}^*} f(\tilde{\theta}) |\det Ad_g^\sharp| e^{-\frac{i}{\hbar} \langle \tilde{\theta}, Ad_{g^{-1}} v \rangle} \frac{d\tilde{\theta}}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= |\det Ad_g^\sharp| \mathcal{F}_\hbar(f)(Ad_{g^{-1}} v) \end{aligned}$$

where $\tilde{\theta} = Ad_{g^{-1}}^\sharp \theta$ and where we used $\langle Ad_g^\sharp \tilde{\theta}, v \rangle = \langle \tilde{\theta}, Ad_{g^{-1}} v \rangle$ for all $g \in G$, $\tilde{\theta} \in \mathcal{G}^*$ and $v \in \mathcal{G}$. \square

Moreover

Lemma 37. *Given a G -system a , and for all $f \in C^\infty(\mathcal{G}^*)$ and $g \in G$, the following formula holds true*

$$J^a(Ad_g^\# f)(x, \xi) = e^{-\frac{i}{\hbar} \langle \xi, x \rangle} \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) a_{g \exp(v)g^{-1}}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{g \exp(-v)g^{-1}}(x) \rangle} \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}}.$$

Proof. Also in this case the proof of the statement follows from a direct computation. Using Lemma 36 to compute $I = e^{+\frac{i}{\hbar} \langle \xi, x \rangle} J^a(Ad_g^\# f)(x, \xi)$, we obtain

$$\begin{aligned} I &= \int_{\mathcal{G}} \mathcal{F}_\hbar(Ad_g^\# f)(v) a_{\exp(v)}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{\exp(-v)}(x) \rangle} \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= \int_{\mathcal{G}} |\det Ad_g^\#| \mathcal{F}_\hbar f(Ad_{g^{-1}} v) a_{\exp(v)}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{\exp(-v)}(x) \rangle} \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= \int_{\mathcal{G}} |\det Ad_g^\#| |\det Ad_g| \mathcal{F}_\hbar f(\tilde{v}) a_{g \exp(\tilde{v})g^{-1}}(x, \xi) e^{\frac{i}{\hbar} \langle \xi, \varphi_{g \exp(-v)g^{-1}}(x) \rangle} \frac{d\tilde{v}}{(2\pi\hbar)^{\frac{n}{2}}}, \end{aligned}$$

where we used the change of variable $\tilde{v} = Ad_{g^{-1}}(v)$ and the identities

$$\exp(Ad_g(v)) = g \exp(v) g^{-1} \quad \text{and} \quad |\det Ad_g^\#| |\det Ad_g| = 1,$$

which hold for every $v \in \mathcal{G}$ and $g \in G$. \square

We can now conclude the proof of Theorem 33. Using Lemma 37 to compute $K = \text{Op}(J^a(Ad_g^\# f))$, we obtain

$$\begin{aligned} (K\psi)(x) &= \int_{\mathbb{R}^{2d}} \psi(\bar{x}) J^a(Ad_g^\# f)(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, (x - \bar{x}) \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}} \\ &= \int_{\mathcal{G}} \int_{\mathbb{R}^{2d}} \psi(\bar{x}) \mathcal{F}_\hbar f(v) a_{g \exp(-v)g^{-1}}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{g \exp(-v)g^{-1}}(x) - \bar{x} \rangle} \frac{dv d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d+n}{2}}} \\ &= \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) (T_{g \exp(v)g^{-1}}^a \psi)(x) \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}}. \end{aligned}$$

Since a is a formal G -system, we have

$$(T_{g \exp(v)g^{-1}}^a \psi)(x) = (T_g^a T_{\exp(v)}^a T_{g^{-1}}^a \psi)(x)$$

for all $g \in G$, $v \in \mathcal{G}$ and $\psi \in C^\infty(\mathbb{R}^d)$, and thus that $K = T_g^a L T_{g^{-1}}$, where L is the operator defined by

$$L = \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) T_{\exp(v)}^a \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}}.$$

Let us now compute its action on functions:

$$\begin{aligned} (L\psi)(x) &= \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) T_{\exp(v)}^a \psi(x) \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \\ &= \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) \left[\int_{\mathbb{R}^{2d}} \psi(\bar{x}) a_{\exp(v)}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{\exp(-v)}(x) - \bar{x} \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}} \right] \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \end{aligned}$$

Unwrapping the Fourier transform in this last expression, we obtain

$$\int_{\mathbb{R}^{2d}} \psi(\bar{x}) \left[e^{-\frac{i}{\hbar} \langle \bar{\xi}, x \rangle} \int_{\mathcal{G}} \mathcal{F}_\hbar f(v) a_{\exp(v)}(x, \bar{\xi}) e^{\frac{i}{\hbar} \langle \bar{\xi}, \varphi_{\exp(-v)}(x) \rangle} \frac{dv}{(2\pi\hbar)^{\frac{n}{2}}} \right] e^{\frac{i}{\hbar} \langle \bar{\xi}, (x - \bar{x}) \rangle} \frac{d\bar{x} d\bar{\xi}}{(2\pi\hbar)^{\frac{d}{2}}},$$

which we recognize to be $\text{Op}(J^a(f))\psi(x)$. Thus $K = T_g^a L T_{g^{-1}}$ is exactly what we wanted to prove: namely,

$$\text{Op}(J^a(Ad_g^\# f)) = T_g^a \text{Op}(J^a(f)) T_{g^{-1}}^a.$$

REFERENCES

- [1] M. Bayen, M. Flato, C. Fronsdal, A., Lichnerowicz, and D. Sternheimer, Deformation theory and quantization I and II, *Ann. Phys.* **111** (1977), 61–151 .
- [2] N. Ben Amar, A comparison between Rieffel’s and Kontsevich’s deformation quantizations for linear Poisson tensors, *Pacific J. of Math.* **229** (2007).
- [3] H. Bursztyn, Momentum maps, dual pairs and reduction in deformation quantization, <http://www.math.berkeley.edu/~alanw/277papers00/bursztyn.pdf>.
- [4] T. Johnson-Freyd, The formal path integral and quantum mechanics, *J. Math. Phys.* **51** (2010).
- [5] A.S. Cattaneo, B. Dherin, and A. Weinstein, Integration of Lie algebroid comorphisms, eprint arXiv:1210.4443.
- [6] A.S. Cattaneo, B. Dherin, and A. Weinstein, Symplectic microgeometry I: micromorphisms, *J. Symplectic Geom.* **8** (2010), no. 2, 205–223.
- [7] A.S. Cattaneo, B. Dherin, and A. Weinstein, Symplectic microgeometry II: generating functions, *Bull. Braz. Math. Society* **4** (2011), 507–536.
- [8] A. S. Cattaneo, B. Dherin and Alan Weinstein, Symplectic microgeometry III: monoids, to appear in *J. Symplectic Geom.*
- [9] A. S. Cattaneo, B. Dherin and Alan Weinstein, Symplectic microgeometry IV: quantization, in preparation.
- [10] B. Dherin and I. Mencattini, Quantization of (volume-preserving) actions on \mathbb{R}^d , eprint, arXiv1202.0886.
- [11] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics **138**, AMS.
- [12] B. Fedosov, Non-abelian reduction in deformation quantization. *Lett. Math. Phys.* **43** (1998), 137–154.
- [13] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67** (1982), 515–538.
- [14] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, (1990).
- [15] S. Gutt, An explicit \ast -product on the cotangent bundle of a Lie group, *Lett. in Math. Phys.* **7** (1983), 249–258,
- [16] K. Hamachi, Differentiability of quantum moment maps and G -invariant star products, *Pacific J. of Math.* **216** (2004).
- [17] A. A. Kirillov, The orbit method I. Geometric quantization, *Representation Theory of Groups and Algebras*, Contemp. Math. (1993) **145**, 1–32 .
- [18] B. Kostant, *Quantization and Unitary Representations*, Lect. Notes in Math. **170** (1970), 87–208.
- [19] L. P. Landsman and M. Rieffel, Induction as generalized Marsden-Weinstein reduction, *J. of Geom. and Phys.* **15** (1995), 285–319.
- [20] L. P. Landsman, *Mathematical Topics between Classical and Quantum Mechanics*, Springer Monographs in Math., Springer-Verlag (1998).
- [21] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis*, Springer-Verlag (2001).
- [22] M. A. Rieffel, Deformation quantization for actions of \mathbb{R}^d , *Mem. Amer. Math. Soc.* **106** (1993).
- [23] M. A. Rieffel, Deformation quantization of Heisenberg manifolds, *Comm. Math. Phys.* **122** (1989), 531–562.
- [24] J. H. Lu, Momentum maps at the quantum level, *Comm. Math. Phys.* **157** (1993), 389–404.
- [25] P. Xu, Fedosov \ast -products and quantum momentum maps, *Comm. Math. Phys.* **197** (1998), 167–197.

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